13. Equilibrium

In this chapter we will combine many of the principles that we have learned in the last few chapters on both linear and rotational motion. The previous chapters were mostly about changes during these types of motion, caused by forces, and the resulting transfer of energy. What, if there are no changes and consequently no transfer of energy? What are the conditions that none of these changes take place? We are talking about static structures that remain in place. This is a very important aspect for engineers designing buildings and other large structures.

13.1 Equilibrium Conditions

Let’s start with a thought question based on a few situations that we have already encountered along the way. What do the following situations have in common?

• a ball being at rest on the bottom of a hole

• a space probe flying through interstellar space at constant velocity after the thrusters have been cut

• a hockey puck sliding on frictionless ice with constant velocity

• a satellite floating at constant velocity and spinning at a constant rate

• a book standing on the bookshelf, supported by bookends

All of the common conditions in these cases can be summarized under the umbrella of two conservation laws.

1. The linear momentum \( \vec{P} \) of the objects is constant and does not change. 
2. The angular momentum \( \vec{L} \) of the objects about their center of mass is constant and does not change.

Objects, for which both conditions

\[ \vec{P} = \text{constant} \quad \text{and} \quad \vec{L} = \text{constant} \quad \text{or} \quad \text{dP/dt = 0 and dL/dt = 0} \]

are met, are in equilibrium. Note that even a moving object may be in equilibrium, if the motion is at constant velocity (magnitude and direction remain constant) and/or at constant angular velocity (magnitude and orientation remain constant).

In the following we will restrict ourselves to static equilibrium, i.e. both \( \vec{P} \) and \( \vec{L} \) are even 0. The objects are not moving in our very reference frame that we are observing in. This additional restriction throws out the sliding puck, the space probe and the spinning satellite. Only the ball and the book are truly in static equilibrium.

In this sense all buildings, bridges, rocks (that are not just in a rockslide) must also be in equilibrium. As we can see from a rockslide, static equilibrium may sometimes be disturbed so that it doesn't hold anymore. Apparently, some structures are more stable than others.

From previous analysis, we have seen that forces acting on objects in equilibrium must be balanced in order to allow \( \text{dP/dt = 0} \) and \( \text{dL/dt = 0} \). Now we know from experience that things may break, because they are not strong enough. Necessary support forces exceed their maximum. This is a very important field in mechanical engineering. For every building, bridge, car, plane, ship etc. engineers must analyze forces on their structure and make sure that limiting loads on the materials are not exceeded.
13.2 Requirements for Equilibrium

As we have learned from Newton's 2nd law and 1st law for linear motion, the condition \( \vec{P} = \) constant means that
\[
\frac{d\vec{P}}{dt} = \vec{F}_{\text{net}} = 0
\]
13.2-1
Translational equilibrium means balance of all forces that are acting on our object, not necessarily that all forces are 0. Only the net force needs to be 0.

Likewise, rotational motion is governed by torque and Newton's 2nd law and 1st law for angular motion. Consequently, the condition \( \vec{L} = \) constant means that
\[
\frac{d\vec{L}}{dt} = \vec{M}_{\text{net}} = 0
\]
13.2-2
Rotational equilibrium means balance of all torques that are acting on our object, not necessarily that all torques are 0. Only the net torque needs to be 0.

In essence:
1. The vector sum of all external forces on the body must be 0.
2. The vector sum of all external torques on the body, measured about any point, must be 0.

These requirements hold for static equilibrium, but they also hold for a more general equilibrium, when objects are in motion, with \( \vec{P} \) and \( \vec{L} \) constant but not 0.

As both equations (13.2-1 and –2) are vector equations, this requires that every one of the three components of the forces and of the torques combine to net components that are 0. We will only consider problems, in which all contributing forces act in the same plane. We make this plane the x-y plane. This also means that any forces that conspire to a torque that wants to rotate the object can only do so about an axis that is perpendicular to the x-y plane. Therefore, the resulting torque must be along the z-axis. As a consequence we will deal only with the following combination of requirements:
\[
\begin{align*}
\vec{F}_{\text{net}, x} &= 0, & \vec{F}_{\text{net}, y} &= 0 \\
\vec{M}_{\text{net}, z} &= 0
\end{align*}
\]
13.2-3
13.2-4
All net forces in x and y sum up to 0 and so do all torques about the z-axis or any axis parallel to the z-axis. These conditions apply to all our examples. However, static equilibrium requires that both \( \vec{P} \) and \( \vec{L} \) are also 0. Checkpoint 13.1

13.3 Center of Gravity

We have already encountered such equilibrium in Chapter 8. The billiard ball on the bottom of the ball is in the state of stable static equilibrium, i.e. it returns to the equilibrium position, after being slightly displaced. In any of the positions slightly displaced from the static equilibrium position, gravity exerts a force on the billiard ball to pull it back. Thus at none of the other positions the ball can be in equilibrium, but it always returns. This is different, if try to place the billiard ball on the other side of the bowl. It is much harder to place the ball such that it does not move, and when we displace the ball, it rolls off rather than to return. We call the latter an unstable static equilibrium.

These examples force us to consider many static situations as to whether they are stable or unstable. Even if a situation seems to be stable for some time, it may become unstable if disturbed enough. Slide “Balancing Rock”
Looking at the “Balancing Rock”, we can literally see it toppling over, if enough of the rock underneath has been eroded or if the rock were to receive enough of a kick. Such situations are

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similar to that of a domino or a toy building block that can be set up on different sides. In some arrangements they are very stable, in others they are only marginally stable. If on edge, tilting the block by a tiny bit is enough to make it fall over. (Fig. 13-2) The difference is, whether the gravitational force can be balanced by the normal force on the supporting edge of the block, or not. Since gravity acts on the center of mass of an object, its location relative to the supporting edge is important.

Let us analyze this situation in some more detail. The gravitational force on an extended object is the sum of all forces on each individual element of the object. However, if we are only interested in the gravitational force on the object as a whole, we can simplify the analysis. The gravitational force relevant to the object is pulling “effectively” with a magnitude according to the total mass of the object on a single point that we call the center of gravity of the object. This statement means that applying the individual forces on each point of the object is not any different from applying the total force just on the center of gravity, neither the net force nor the net torque on the object will be different. Until now we have tacitly assumed that the gravitational force \( \vec{F}_g \) acts on the center of mass of an object. Together with our new statement this is equivalent to assuming that the center of mass and the center of gravity are identical. We will now show that this is in fact a true statement. The gravitational force \( \vec{F}_g \) on an object with mass \( M \) is equal to \( MG \hat{g} \), where \( \hat{g} \) is the acceleration that the force would produce, if the object fell freely. Thus we have to show: If \( \hat{g} \) is the same for all elements of an object, then the center of gravity (CoG) is identical with the center of mass (CoM).

This is correct (to very high precision) for everyday objects on Earth, because over the extent of the objects Earth’s gravity varies very little. If we look at a moon that is very close to a planet relative to the planet’s diameter, such as Io to Jupiter, we cannot readily use this assumption anymore.

Let us analyze the result of gravity on individual elements of an extended object (Fig. 13-4a). For \( m_i \) we get the magnitude of the gravitational force \( F_{gi} = mg_i \), because, in principle, \( g \) may be different at different locations. The force also produces a torque \( \vec{t} \) on each element about the origin O, with the moment arm \( x_i \).

\[
\vec{t} = x_i F_{gi}
\]

13.3-1

The net torque on the entire object is then

\[
\vec{t}_{net} = \sum \vec{t} = \sum x_i F_{gi}
\]

13.3-2

For the object as a whole we defined that the gravitational force \( \vec{F}_g \) (which is the sum of the forces on all elements \( F_{gi} \)) acts on the center of gravity (cog) of the body. Consequently, this force must produce a net torque \( \vec{t} \) on the object about the origin O with moment arm \( x_{CoG} \).

\[
\vec{t} = x_{CoG} F_g = x_{CoG} \sum F_{gi}
\]

13.3-3

According to our definition of the center of gravity the torque of \( \vec{F}_g \) on the CoG must be equal to the net torque due to all individual forces \( F_{gi} \) on the elements of the object. This means 13.3-2 must be equal to 13.3-3.

\[
x_{CoG} \sum F_{gi} = \vec{t} = \vec{t}_{net} = \sum x_i F_{gi}
\]

13.3-4

Using the definition for gravitation we get:

\[
x_{CoG} m_i g_i = \sum x_i m_i g_i
\]

13.3-5

Now we use the approximation that the strength of gravity is the same throughout our object, which is valid within reason for most every objects on Earth, and this is what we are concerned
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13.3-5 simplifies to:

\[ x_{\text{cog}} m_i = x_i m_i. \]

Because the sum of all element masses is \( M \) we retrieve as result the definition of the center of mass (CoM).

\[ x_{\text{cog}} = \frac{1}{M} \sum_i x_i m_i = x_{\text{com}}. \] (Q.e.D.)  

13.3-7

13.4 Examples

Sample Problem 13-2
Sample Problem 13-3 (Watch Homework)
Do CP `13-4 as problem step by step
Uniform Sphere on Incline

Checkpoint 13-2

Demo Apple on Skewer

Checkpoint 13-4 (HW)